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Kerr-Schild Structure and Harmonic 2-forms on (A)dS-Kerr-NUT Metrics

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ABSTRACT

We demonstrate that the general (A)dS-Kerr-NUT solutions in D dimensions with $([D/2], [(D+1)/2])$ signature admit $[D/2]$ linearly-independent, mutually-orthogonal and affinely-parameterised null geodesic congruences. This enables us to write the metrics in a multi-Kerr-Schild form, where the mass and all of the NUT parameters enter the metrics linearly. In the case of $D = 2n$, we also obtain n harmonic 2-forms, which can be viewed as charged (A)dS-Kerr-NUT solution at the linear level of small-charge expansion, for the higher-dimensional Einstein-Maxwell theories. In the BPS limit, these 2-forms reduce to $n-1$ linearly-independent ones, whilst the resulting Calabi-Yau metric acquires a Kähler 2-form, leaving the total number the same.

1 Introduction

One intriguing feature of General Relativity is that, despite its high degree of non-linearity, many exact solutions can be cast into a Kerr-Schild form [1] where non-trivial parameters such as mass, charge, or cosmological constant enter the metrics as a linear perturbation of flat spacetime. A simple example is the (A)dS metric, which can be written as

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_n^2 + \Lambda\, r^2 (dt - dr)^2\,, \tag{1}$$

where the first three terms describe the $(n+2)$-dimensional Minkowski spacetime and the cosmological constant enters the last term linearly. More complicated examples include the Plebanski metric [2]; in (2,2) signature, the Plebanski metric can have a double Kerr-Schild form where both the mass and the NUT charge enter the metric linearly [3].

The most general higher-dimensional (A)dS-Kerr-NUT solutions, which can be viewed as higher-dimensional generalisations of the Plebanski metric, were recently obtained in [4]. The solutions are parameterised by the mass, multiple NUT charges and arbitrary orthogonal rotations. The metrics have $U(1)^n$ isometries, where $n = [(D+1)/2]$. They are demonstrated [5] to be of type D in the higher-dimensional generalisation [6] of the Petrov classification.

Many further interesting properties of the metrics were obtained, such as the separability of the Hamiltonian-Jacobi and Klein-Gordon equations [7], and the existence of Killing-Yano tensors [8]. The metrics also admit BPS limits where the Killing spinors can emerge [4]. In the odd $2n+1$ dimensions, this leads to a large class of Einstein-Sasaki metrics with $U(1)^n$ isometry, generalising the previously known $Y^{p,q}$ [9] and L^{pqr} [10] spaces. In the even $2n$ dimensions, this leads to the non-compact Calabi-Yau metrics that can provide a resolution of the cone over the Einstein-Sasaki metrics constructed in the odd dimensions [11, 12].

In this letter, we demonstrate in section 2 that the D-dimensional (A)dS-Kerr-NUT solution admits $[D/2]$ linearly-independent, mutually-orthogonal and affinely parameterised null geodesic congruences upon Wick-rotation of the metric to $([D/2], [(D+1)/2])$ signature. This enables us to cast the metric into the multi-Kerr-Schild form, where the mass and all of the NUT parameters enter the metric linearly. In section 3, we obtain n harmonic 2-forms on the (A)dS-Kerr-NUT metrics in $D = 2n$ dimensions. In the BPS limit, these n harmonic 2-forms becomes linearly dependent, and the number of linearly-independent ones becomes $n-1$. However, a Kähler 2-form emerges under the BPS limit, and hence the total number of harmonic 2-forms remains n. We conclude the letter in section 4.

2 Multi-Kerr-Schild structure

Let us first consider the case of $D = 2n+1$ dimensions, for which the metric was given in [4]. In order to put the metric in a Kerr-Schild form, it is necessary to Wick rotate to $(n, n+1)$ signature. This can be easily achieved by Wick rotating all the spatial $U(1)$ coordinates. The corresponding metric is then given by

$$ds^2 = \sum_{\mu=1}^{n} \Big\{ \frac{dx_\mu^2}{Q_\mu} - Q_\mu \Big(\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\psi_k \Big)^2 \Big\} + \frac{c}{(\prod_{\nu=1}^{n} x_\nu^2)} \Big(\sum_{k=0}^{n} A^{(k)}\, d\psi_k \Big)^2 \,, \tag{2}$$

where

$$\begin{aligned} Q_\mu &= \frac{X_\mu}{U_\mu}\,, \qquad U_\mu = {\prod}'^{n}_{\nu=1} (x_\nu^2 - x_\mu^2)\,, \qquad X_\mu = \sum_{k=1}^{n} c_k\, x_\mu^{2k} + \frac{c}{x_\mu^2} - 2b_\mu\,, \\ A_\mu^{(k)} &= \sum_{\nu_1<\nu_2<\cdots<\nu_k}{}' x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2\,, \qquad A^{(k)} = \sum_{\nu_1<\nu_2\cdots<\nu_k} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2\,, \end{aligned} \tag{3}$$

The prime on the summation and product symbols in the definition of $A_\mu^{(k)}$ and U_μ indicates that the index value μ is omitted in the summations of the ν indices over the range $[1, n]$. Note that ψ_0 was denoted as t in [4], playing the rôle of the time like coordinate in the $(1, 2n)$ spacetime signature. In this way of writing the metric, all of the integration constants of the solution enter only in the functions X_μ. The constant $c_n = (-1)^n\Lambda$ is fixed by the value of the cosmological constant, with $R_{\mu\nu} = 2n\Lambda\, g_{\mu\nu}$. The other $2n$ constants c_k, c and b_μ are arbitrary. These are related to the n rotation parameters, the mass and the $(n-1)$ NUT parameters, with one parameter being trivial and removable through a scaling symmetry [4]. Note that in $(n, n+1)$ signature, the NUT charges are really masses with respect to different time-like Killing vectors. However, we shall continue to refer them as NUT charges.

We now re-arrange the metric (2) into the form

$$\begin{aligned} ds^2 &= -\sum_{\mu=1}^{n} \frac{X_\mu}{U_\mu} \Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\psi_k + \frac{U_\mu}{X_\mu} dx_\mu \Big] \Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\psi_k - \frac{U_\mu}{X_\mu} dx_\mu \Big] \\ &\quad + \frac{c}{(\prod_{\nu=1}^{n} x_\nu^2)} \Big(\sum_{k=0}^{n} A^{(k)}\, d\psi_k \Big)^2 \,. \end{aligned} \tag{4}$$

If we perform the following coordinate transformation,

$$d\hat\psi_k = d\psi_k + \sum_{\mu=1}^{n} \frac{(-x_\mu^2)^{n-k-1}}{X_\mu} dx_\mu\,, \qquad k = 0, \cdots, n\,, \tag{5}$$

the metric can then be cast into the n-Kerr-Schild form, namely

$$ds^2 = d\bar s^2 + \sum_{\mu=1}^{n} \frac{2b_\mu}{U_\mu} \Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\hat\psi_k \Big]^2 \,, \tag{6}$$

where

$$
\begin{aligned}
d\bar{s}^2 &= -\sum_{\mu=1}^{n}\Big\{\frac{\bar{X}_\mu}{U_\mu}\Big[\sum_{k=0}^{n-1}A_\mu^{(k)}\,d\hat{\psi}_k\Big]^2-2\Big[\sum_{k=0}^{n-1}A_\mu^{(k)}\,d\hat{\psi}_k\Big]dx_\mu\Big\}\\
&\quad+\frac{c}{(\prod_{\nu=1}^{n}x_\nu^2)}\Big(\sum_{k=0}^{n}A^{(k)}\,d\hat{\psi}_k\Big)^2\,,\\
\bar{X}_\mu &= \sum_{k=1}^{n}c_k\,x_\mu^{2k}+\frac{c}{x_\mu^2}\,.
\end{aligned}
\tag{7}
$$

It is straightforward to verify that the metric $d\bar{s}^2$ is that of pure (A)dS spacetime. The mass and NUT parameters b_μ appear linearly in the metric ds^2. It should be emphasised that although the constants c and c_k with $k<n$ are trivial in the metric $d\bar{s}^2$, they provide non-trivial angular momentum parameters in the metric ds^2. It is interesting to note that all of the constants c_k, including c_n that is related to the cosmological constant, appear linearly in the metric, and can all be extracted from $d\bar{s}^2$ and grouped in the second term of (6). This implies that all the parameters, the mass, NUTs and angular momenta and cosmological constant can enter the metric linearly as a perturbation of flat spacetime. In this letter, we shall consider in detail only the Kerr-Schild form where the mass and NUT parameters enter the metric linearly as a perturbation of pure (A)dS spacetime.

The (A)dS metric (7) can be diagonalised, in a way that the second term of (6) remains simple. To do so, let us first rewrite the $\bar{X}_\mu$ as follows

$$
\bar{X}_\mu=\frac{(1+\Lambda\,x_\mu^2)}{x_\mu^2}\prod_{k-1}^{n}(a_k^2-x_\mu^2)\,. \tag{8}
$$

Then we complete the square in $d\bar{s}^2$:

$$
d\bar{s}^2=\sum_{\mu=1}^{n}\Big\{\frac{U_\mu}{\bar{X}_\mu}dx_\mu^2-\frac{\bar{X}_\mu}{U_\mu}\Big[\sum_{k=0}^{n-1}A_\mu^{(k)}\,d\hat{\psi}_k-\frac{U_\mu}{\bar{X}_\mu}dx_\mu\Big]^2\Big\}+\frac{c}{(\prod_{\nu=1}^{n}x_\nu^2)}\Big(\sum_{k=0}^{n}A^{(k)}\,d\hat{\psi}_k\Big)^2, \tag{9}
$$

and make the coordinate transformation,

$$
d\tilde{\psi}_k=-d\hat{\psi}_k+\sum_{\mu=1}^{n}\frac{(-x_\mu^2)^{n-k-1}}{\bar{X}_\mu}dx_\mu\,,\qquad k\;=\;0\,,\;\cdots\;,n\,. \tag{10}
$$

The metric can be put into a new form,

$$
ds^2=d\bar{s}^2+\sum_{\mu=1}^{n}\frac{2b_\mu}{U_\mu}\Big[\sum_{k=0}^{n-1}A_\mu^{(k)}\,d\tilde{\psi}_k-\frac{U_\mu}{\bar{X}_\mu}dx_\mu\Big]^2\,, \tag{11}
$$

where

$$
d\bar{s}^2=\sum_{\mu=1}^{n}\Big\{\frac{U_\mu}{\bar{X}_\mu}dx_\mu^2-\frac{\bar{X}_\mu}{U_\mu}\Big[\sum_{k=0}^{n-1}A_\mu^{(k)}\,d\tilde{\psi}_k\Big]^2\Big\}+\frac{c}{(\prod_{\nu=1}^{n}x_\nu^2)}\Big(\sum_{k=0}^{n}A^{(k)}\,d\tilde{\psi}_k\Big)^2\,. \tag{12}
$$

Performing a recombination of the $U(1)$ coordinates, namely

$$\tau = \sum_{k=0}^{n} B^{(k)}\, d\tilde{\psi}_k\,, \qquad \frac{\varphi_i}{a_i} = \sum_{k=1}^{n} B_i^{(k-1)} d\tilde{\psi}_k - \Lambda \sum_{k=0}^{n-1} B_i^{(k)}\, d\tilde{\psi}_k\,, \qquad i\,=\,1\,,\,\cdots\,,n\,, \tag{13}$$

where

$$B_i^{(k)} = \sideset{}{'}\sum_{j_1<j_2<\cdots<j_k} a_{j_1}^2 a_{j_2}^2 \cdots a_{j_k}^2\,, \qquad B^{(k)} = \sum_{j_1<j_2\cdots<j_k} a_{j_1}^2 a_{j_2}^2 \cdots a_{j_k}^2\,, \tag{14}$$

the odd dimensional (A)dS-Kerr-NUT metrics can be expressed as

$$ds^2 = d\bar{s}^2 + \sum_{\mu=1}^{n} \frac{2b_\mu}{U_\mu} (k_{(\mu)\alpha} dx^\alpha)^2\,, \tag{15}$$

$$d\bar{s}^2 = \frac{W}{\prod_{i=1}^n \Xi_i} d\tau^2 + \sum_{\mu=1}^{n} \frac{U_\mu}{\bar{X}_\mu}\, dx_\mu^2 - \sum_{i=1}^{n} \frac{\gamma_i}{\Xi_i {\prod'}_{k=1}^{n} (a_i^2 - a_k^2)} d\varphi_i^2\,, \tag{16}$$

$$k_{(\mu)\alpha} dx^\alpha = \frac{W}{1+\Lambda\, x_\mu^2} \frac{d\tau}{\prod_{i=1}^n \Xi_i} - \frac{U_\mu\, dx_\mu}{\bar{X}_\mu} - \sum_{i=1}^{n} \frac{a_i\, \gamma_i d\varphi_i}{(a_i^2 - x_\mu^2)\Xi_i {\prod'}_{k=1}^{n} (a_i^2 - a_k^2)}\,, \tag{17}$$

where

$$\Xi_i = 1 + \Lambda\, a_i^2\,, \qquad \gamma_i = \prod_{\nu=1}^{n} (a_i^2 - x_\nu^2)\,, \qquad W = \prod_{\nu=1}^{n} (1 + \Lambda\, x_\nu^2)\,. \tag{18}$$

If we set all but one of the b_μ to zero, the result reduces to the Kerr-Schild form for rotating (A)dS black holes obtained previously in [13].

We now turn our attention to the the case of $D = 2n$ dimensions. The corresponding (A)dS-Kerr-NUT metrics were obtained in [4]. After performing Wick rotations, the metric with (n, n) signature is given by

$$ds^2 = \sum_{\mu=1}^{n} \Big\{ \frac{dx_\mu^2}{Q_\mu} - Q_\mu \Big(\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\psi_k \Big)^2 \Big\}\,, \tag{19}$$

where we Q_μ, U_μ and $A_\mu^{(k)}$ have the same form as those in the even dimensions, given in (3). The functions X_μ are given by

$$X_\mu = \sum_{k=0}^{n} c_k\, x_\mu^{2k} + 2b_\mu\, x_\mu\,. \tag{20}$$

The constants c_k and b_μ are arbitrary, except for $c_n = (-1)^n \Lambda$, which is fixed by the value of the cosmological constant, $R_{\mu\nu} = (2n-1)\Lambda\, g_{\mu\nu}$. The metric can be re-arranged into the form

$$ds^2 = -\sum_{\mu=1}^{n} \frac{X_\mu}{U_\mu} \Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\psi_k + \frac{U_\mu}{X_\mu} dx_\mu \Big] \Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\psi_k - \frac{U_\mu}{X_\mu} dx_\mu \Big]\,. \tag{21}$$

After performing the coordinate transformation

$$d\hat{\psi}_k = d\psi_k + \sum_{\mu=1}^{n} \frac{(-x_\mu^2)^{n-k-1}}{X_\mu} dx_\mu \,, \qquad k \;=\; 0\,,\; \cdots\;, n-1\,, \tag{22}$$

the metric can be cast into the n-Kerr-Schild form,

$$ds^2 = d\bar{s}^2 - \sum_{\mu=1}^{n} \frac{2b_\mu x_\mu}{U_\mu} \Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\hat{\psi}_k\Big]^2 \tag{23}$$

where

$$\begin{aligned} d\bar{s}^2 &= -\sum_{\mu=1}^{n} \Big\{ \frac{\bar{X}_\mu}{U_\mu} \Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\hat{\psi}_k\Big]^2 - 2\Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\hat{\psi}_k\Big] dx_\mu \Big\}\,, \\ \bar{X}_\mu &= \sum_{k=0}^{n} c_k\, x_\mu^{2k}\,. \end{aligned} \tag{24}$$

It is straightforward to verify that $d\bar{s}^2$ is the metric for pure (A)dS spacetime. As in the odd dimensions, this metric can be put into a diagonal form, while keeping the second term of (23) simple. To do that, we first reparameterise X_μ as

$$\bar{X}_\mu = -(1 - g^2 x_\mu^2) \prod_{k=1}^{n-1} (a_k^2 - x_\mu^2)\,. \tag{25}$$

We then complete the square in $d\bar{s}^2$, *i.e.*

$$d\bar{s}^2 = \sum_{\mu=1}^{n} \Big\{ \frac{U_\mu}{\bar{X}_\mu} dx_\mu^2 - \frac{\bar{X}_\mu}{U_\mu} \Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\hat{\psi}_k - \frac{U_\mu}{\bar{X}_\mu} dx_\mu\Big]^2 \Big\} \tag{26}$$

and make the coordinate transformation

$$d\tilde{\psi}_k = -d\hat{\psi}_k + \sum_{\mu=1}^{n} \frac{(-x_\mu^2)^{n-k-1}}{\bar{X}_\mu} dx_\mu \,, \qquad k \;=\; 0\,,\; \cdots\;, n-1\,. \tag{27}$$

The metric (23) can then be put into a new form:

$$ds^2 = d\bar{s}^2 - \sum_{\mu=1}^{n} \frac{2b_\mu x_\mu}{U_\mu} \Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\tilde{\psi}_k - \frac{U_\mu}{\bar{X}_\mu} dx_\mu\Big]^2\,, \tag{28}$$

where

$$d\bar{s}^2 = \sum_{\mu=1}^{n} \Big\{ \frac{U_\mu}{\bar{X}_\mu} dx_\mu^2 - \frac{\bar{X}_\mu}{U_\mu} \Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\tilde{\psi}_k\Big]^2 \Big\}\,. \tag{29}$$

The $d\bar{s}^2$ metric can now straightforwardly be diagonalised by means of the coordinate transformation

$$\tau = \sum_{k=0}^{n-1} B^{(k)}\, d\tilde{\psi}_k \,, \qquad \frac{\varphi_i}{a_i} = \sum_{k=1}^{n-1} B_i^{(k-1)} d\tilde{\psi}_k + g^2 \sum_{k=0}^{n-2} B_i^{(k)}\, d\tilde{\psi}_k \qquad i \;=\; 1\,,\; \cdots\;, n-1\,, \tag{30}$$

where

$$B_i^{(k)} = \sideset{}{'}\sum_{j_1<j_2<\cdots<j_k} a_{j_1}^2 a_{j_2}^2 \cdots a_{j_k}^2\,, \qquad B^{(k)} = \sum_{j_1<j_2\cdots<j_k} a_{j_1}^2 a_{j_2}^2 \cdots a_{j_k}^2\,. \tag{31}$$

The even dimensional (A)-dS Kerr-NUT metrics can now be expressed as

$$ds^2 = d\bar{s}^2 - \sum_{\mu=1}^{n} \frac{2b_\mu x_\mu}{U_\mu} (k_{(\mu)\alpha} dx^\alpha)^2\,, \tag{32}$$

where

$$\begin{aligned} d\bar{s}^2 &= \frac{W}{\prod_{i=1}^{n-1}\Xi_i}\, d\tau^2 + \sum_{\mu=1}^{n} \frac{U_\mu}{\bar{X}_\mu}\, dx_\mu^2 - \sum_{i=1}^{n-1} \frac{\gamma_i}{a_i^2 \Xi_i \prod'^{\,n-1}_{k=1}(a_i^2 - a_k^2)} d\varphi_i^2\,, \qquad (33)\\ k_{(\mu)\alpha} dx^\alpha &= \frac{W}{1 - g^2 x_\mu^2} \frac{d\tau}{\prod_{i=1}^{n-1}\Xi_i} - \frac{U_\mu dx_\mu}{\bar{X}_\mu} - \sum_{i=1}^{n-1} \frac{\gamma_i d\varphi_i}{(a_i^2 - x_\mu^2) a_i \Xi_i \prod'^{\,n-1}_{k=1}(a_i^2 - a_k^2)}\,, \qquad (34) \end{aligned}$$

where Ξ_i, γ_i and W have the same structure as that in the even dimensions, given by (18). When all but one of the b_μ vanishes, the metric reduces to the Kerr-Schild form of the rotating (A)dS black hole obtained in [13].

To summarise, we find that in both even and odd dimensions, the (A)dS-Kerr-NUT solution can be cast into the following multi-Kerr-Schild form:

$$ds^2 = d\bar{s}^2 + \sum_{\mu=1}^{n} \frac{2b_\mu\, f(x_\mu)}{U_\mu} (k_{(\mu)\alpha} dx^\alpha)^2\,, \tag{35}$$

where $f(x_\mu) = 1$ for odd dimensions and $f(x_\mu) = x_\mu$ for even dimensions. The vectors $k_{(\mu)\alpha}$ are n linearly-independent, mutually-orthogonal and affinely-parameterised null geodesic congruences, satisfying

$$k_{(\mu)\alpha} k^\alpha_{(\nu)} = 0\,, \qquad k^\alpha_{(\mu)} \bar{\nabla}_\alpha k_{(\mu)\beta} = 0\,. \tag{36}$$

Note that the index α in $k_{\alpha(\mu)}$ can be raised with either $g^{\alpha\beta}$ or $\bar{g}^{\alpha\beta}$ for the above conditions to be satisfied.

3 Harmonic 2-forms in $D = 2n$ dimensions

In this section, we find n harmonic 2-forms $G^{(\mu)}_{(2)} = dB^{(\mu)}_{(1)}$ on the $2n$-dimensional (A)dS-Kerr-NUT metric (19), where we use the index $\mu = 1, 2, \ldots n$ to label the harmonic 2-forms. The potentials have a rather simple form, given by

$$B^{(\mu)}_{(1)} = \frac{x_\mu}{U_\mu} \Big(\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\psi_k \Big)\,. \tag{37}$$

The metric (19) admits a natural vielbein basis, namely

$$e^\mu = \frac{dx_\mu}{\sqrt{Q_\mu}}\,, \qquad \tilde e^\mu = \sqrt{Q_\mu}\Big(\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\psi_k\Big)\,. \tag{38}$$

In this vielbein basis, the harmonic 2-forms $G_{(2)}^{(\mu)}$ are given by

$$G_{(2)}^{(\mu)} = \sum f_\nu^{(\mu)}\, e^\nu \wedge \tilde e^\nu\,, \tag{39}$$

where the coefficients are

$$\begin{aligned} f_\mu^{(\mu)} &= \frac{1}{U_\mu^2}\Big[A^{(n-1)} + \sum_{k=1}^{n-2}(-1)^k(2k+1)x_\mu^{2(k+1)}A_\mu^{(n-k-2)}\Big]\,,\\ f_\nu^{(\mu)} &= -\frac{2x_\mu x_\nu}{U_\mu^2}\prod_{\rho\neq\mu,\nu}(x_\rho^2 - x_\mu^2)\,, \qquad \text{with } \mu\neq\nu\,. \end{aligned} \tag{40}$$

We verify with low-lying examples that all of the $G_{(2)}^{(\mu)}$ are harmonic, *i.e.* $dG_{(2)}^{(\mu)} = 0 = d * G_{(2)}^{(\mu)}$. It is worth observing that these 2-forms are harmonic regardless of the detailed structure of the functions X_μ.

It was shown in [4] that the BPS limit of the metric (19) gives rise to the non-compact Calabi-Yau metric that can provide a resolutions of the cone over the Einstein-Sasaki spaces. Under suitable coordinate transformation, the metric is given by

$$ds^2 = \sum_{\mu=1}^{n}\Big\{\frac{dx_\mu^2}{Q_\mu} + Q_\mu\Big(\sum_{k=0}^{n-1}A_\mu^{(k)}\, d\psi_k\Big)^2\Big\}\,, \tag{41}$$

where we define

$$\begin{aligned} Q_\mu &= \frac{4X_\mu}{U_\mu}\,, \qquad U_\mu = {\prod}'^{n}_{\nu=1}(x_\nu - x_\mu)\,, \qquad X_\mu = x_\mu\prod_{k=1}^{n-1}(x_\mu+\alpha_k) + 2b_\mu\,,\\ A_\mu^{(k)} &= {\sum_{\nu_1<\nu_2<\cdots<\nu_k}}' x_{\nu_1}x_{\nu_2}\cdots x_{\nu_k}\,. \end{aligned} \tag{42}$$

Note that we have Wick rotated the metric to have Euclidean signature. We can choose the same form of the vielbein basis as in (38). The Kähler 2-form is then given by

$$J_{(2)} = \sum_{\mu=1}^{n} e^\mu\wedge\tilde e^\mu\,. \tag{43}$$

The 1-form potentials for the harmonic 2-forms are given by

$$B_{(1)}^{(\mu)} = \frac{1}{U_\mu}\Big(\sum_{k=0}^{n-1}A_\mu^{(k)}\, d\psi_k\Big)\,. \tag{44}$$

The corresponding harmonic 2-forms $G_{(2)}^{(\mu)}$ have the same form as in (39), with the functions $f_\nu^{(\mu)}$ are given by

$$f_\nu^{(\mu)} = \frac{2}{U_\mu^2} \prod_{\rho\neq\mu,\nu} (x_\rho - x_\mu)\,,\text{with } \mu\neq\nu\,,\qquad f_\mu^{(\mu)} = -\sum_{\nu\neq\mu} f_\nu^{(\mu)}\,. \tag{45}$$

Note that $G_{(2)}^{(\mu)}$ satisfy the linear relation $\sum_{\mu=1}^n G_{(2)}^{(\mu)} = 0$. Thus, in the BPS limit, there are $(n-1)$ linearly independent such harmonic 2-forms. Together with the Kähler 2-form, the total number of harmonic 2-forms is n again.

4 Conclusion

In this letter, we explicitly express the general (A)dS-Kerr-NUT metrics in Kerr-Schild form for both even and odd dimensions. We demonstrate that, in a suitable coordinate system the mass, NUT and angular momentum parameters enter linearly in the metric, and hence they can be viewed as a linear perturbation of pure (A)dS spacetime.

We also obtain n harmonic 2-forms on the $2n$-dimensional (A)dS-Kerr-NUT metrics. An interesting property of these harmonic 2-forms is that the closure and co-closure do not depend on the detailed structure of the functions X_μ. This provides a potential ansatz for charged (A)dS-Kerr-NUT solutions for pure Einstein-Maxwell theories in higher dimensions, whose explicit analytical solutions remain elusive. In the case of four dimensions, the back-reaction of the gauge field to the Einstein equations gives precisely the charged Plebanski metric [2], where only the functions X_μ in the metric have extra contributions from the electric and magnetic charges. However, the same phenomenon does not occur in higher dimensions; nevertheless, the harmonic 2-forms we constructed can be viewed as charged (A)dS-Kerr-NUT solutions at the linear level for small-charge expansion. Together with the charged slowly-rotating black holes obtained in [14, 15], our results may lead to the general charged (A)dS-Kerr-NUT solutions.

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